Instructions:
2. Work each problem on the exam booklet in the space provided.
3. Write neatly and clearly for partial credit. Cross out any material you do not want graded.

Name: __________________________

Problem 1: _______________________/20
Problem 2: _______________________/20
Problem 3: _______________________/25
Problem 4: _______________________/20
Problem 5: _______________________/15
Total: _______________________/100

Maxwell’s Equations:
Gauss’ Law: \( \phi_{in} = \phi_{out} \)
Ampere’s Law: \( \sum_k H_k l_k = I_{\text{enclosed}} \)
Faraday’s Law: \( v(t) = \frac{d\lambda}{dt} \implies p_m(t) = i(t) \frac{d\lambda}{dt} \)
Energy and Coenergy: \( W_m = \int_0^t p_m(t) \, dt = \int_0^\lambda i \, d\lambda, \quad W'_m = \lambda i - W_m = \int_0^i \lambda \, di \)

Magnetic Circuits:
\( F = Ni = \mathcal{R} \phi \)
\( \lambda = N\phi = NBA = \left( \frac{N^2}{\mathcal{R}} \right) i = Li \)
\( \mathcal{R} = \frac{1}{\mu_r \mu_o A} \)
\( \mu_o = 4\pi \times 10^{-7} \text{ (H/m)} \)
Problem 1 (20 Points)

(a) The given \( \lambda-i \) characteristic is: (i) Linear (ii) **Nonlinear**. (Underline the correct answer.)

(b) Find the magnetic energy \( W_{mA} \) (J) at point A.
\[
W_{mA} = \int_0^1 i \, d\lambda = \frac{1}{2} \times 1 \times 2 = 1 \text{ J}
\]

(c) Find the magnetic coenergy \( W'_{mA} \) (J) at point A.
\[
W'_{mA} = \int_0^2 \lambda \, di = \frac{1}{2} \times 1 \times 2 = 1 \text{ J} = \lambda_A i_A - W_{mA} = 1 \times 2 - 1 = 1 \text{ J}
\]

(d) Find the magnetic energy \( W_{mB} \) (J) at point B.
\[
W_{mB} = \int_0^3 i \, d\lambda = 7 \text{ J}
\]

(e) Find the magnetic coenergy \( W'_{mB} \) (J) at point B.
\[
W'_{mB} = \int_0^4 \lambda \, di = 5 \text{ J} = \lambda_B i_B - W_{mB} = 3 \times 4 - 7 = 5 \text{ J}
\]

(f) Find the magnetic energy \( W_{mC} \) (J) at point C.
\[
W_{mC} = \int_0^4 i \, d\lambda = 12 \text{ J}
\]

(g) Find the magnetic coenergy \( W'_{mC} \) (J) at point C.
\[
W'_{mC} = \int_0^6 \lambda \, di = 12 \text{ J} = \lambda_C i_C - W_{mC} = 4 \times 6 - 12 = 12 \text{ J}
\]
Problem 2 (20 points)

The above magnetic structure has the following dimensions:

\[ d = w = 10 \text{ cm}, \quad g = \pi \text{ mm}, \quad N = 100 \text{ turns}. \]

(a) Find the current \( I \) (A) required to create a magnetic flux density \( B_1 = 1 \text{ T} \) in the center air gap and deduce that \( B_2 = 1 \text{ T} \).

\[
NI = H_1 g = \frac{B_1 g}{\mu_0} \implies I = \frac{B_1 g}{\mu_0 N} = \frac{1 \times \pi \times 10^{-3}}{4\pi \times 10^{-7} \times 100} = 25 \text{ A}
\]

(b) Compute the total energy stored \( W_m \) (J) stored in the two air gaps.

\[
W_m = \left( \frac{1}{2} B_1 H_1 \right) (wdg) + \left( \frac{1}{2} B_2 H_2 \right) (2wdg) = \left( \frac{1}{2} B_1 H_1 \right) (3wdg) = \frac{3wdgB_1^2}{2\mu_0} = 37.5 \text{ J}
\]

(c) Deduce the inductance \( L \) (mH) of this coil.

\[
W_m = \frac{1}{2} LI^2 \implies L = \frac{2W_m}{I^2} = \frac{2 \times 37.5}{25^2} = 0.120 \text{ H} = 120 \text{ mH}
\]

(d) Find the magnetic fluxes \( \phi \) (Wb), \( \phi_1 \) (Wb) and \( \phi_2 \) (Wb) in the left leg, center air gap, and right air gap, respectively.

\[
\phi_1 = B_1 A_1 = B_1 (wd) = 1 \times 0.1 \times 0.1 = 0.01 \text{ Wb}
\]
\[
\phi_2 = B_2 A_2 = B_2 (2wd) = 1 \times 0.2 \times 0.1 = 0.02 \text{ Wb}
\]
\[
\phi = \phi_1 + \phi_2 = 0.01 + 0.02 = 0.03 \text{ Wb}
\]

(e) Find the magnetic flux linkages \( \lambda \) (Wb-t) of the coil and deduce the value of the inductance \( L \) (mH) of this coil using this second approach.

\[
\lambda = N\phi = 100 \times 0.03 = 3 \text{ Wb-t} = LI \implies L = \frac{\lambda}{I} = \frac{3}{25} = 120 \text{ mH}
\]
**Problem 3 (25 Points)**

The dimensions of the above magnetic structure are given in the table below. A current \( I = 23.75 \) A flows in the exciting winding which has 100 turns. Refer to the magnetization curves of both materials and neglect leakage fluxes.

<table>
<thead>
<tr>
<th></th>
<th>Material #1</th>
<th>Material #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean path length</td>
<td>( l_1 = 0.2 ) m</td>
<td>( l_2 = 0.4 ) m</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>( A_1 = 10 ) cm(^2)</td>
<td>( A_2 = 10 ) cm(^2)</td>
</tr>
</tbody>
</table>

Assuming that the magnetic core is saturated, find:

\[
B_1 \ (T), \ H_1 \ (A\cdot t/m), \ B_2 \ (T), \ H_2 \ (A\cdot t/m), \ \phi_1 \ (mWb), \ \phi_2 \ (mWb).
\]

**Solution:**

Material #2 saturates first. Therefore,

\[
\begin{align*}
B_{2,max} & = 1.5 \ T \\
\phi_1 &= B_1 A_1 = \phi_2 = B_2 A_2 \implies B_{1,max} = B_{2,max} = 1.5 \ T \\
\phi_1 &= B_{1,max} A_1 = \phi_2 = B_{2,max} A_2 = 1.5 \times 0.001 = 1.5 \ mWb \\
B_{1,max} & = 1.5 \ T \implies H_1 = 1875 \ A\cdot t/m \\
NI &= H_1 l_1 + H_2 l_2 \\
H_2 &= \frac{NI - H_1 l_1}{l_2} = \frac{100 \times 23.75 - 1875 \times 0.2}{0.4} = 5000 \ A\cdot t/m
\end{align*}
\]
Problem 4 (20 Points)

Two coils are wound as shown on a magnetic core with infinite permeability. These two coils are characterized by the following flux-current relationships:

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} =
\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

Add polarity marks to the two mmfs shown on the equivalent magnetic circuit on the right and choose \(\phi_1\) and \(\phi_2\) in the flux direction pushed by these mmfs. Neglect leakage and fringing effects in this problem.

Solve for \(\phi_1\), \(\phi_2\), \(\lambda_1\), and \(\lambda_2\) and deduce \(L_{11}\), \(L_{12}\), \(L_{21}\), and \(L_{22}\) in terms of \(R_g = g/\mu_0 wd\), \(N_1\) and \(N_2\).

Solution:

\[
\begin{cases}
N_1 i_1 &= R_g (\phi_1 + \phi_2) \\
N_2 i_2 &= R_g (\phi_1 + \phi_2) + R_g \phi_2
\end{cases}
\]

\[
\phi_2 = -\frac{N_1}{R_g} i_1 + \frac{N_2}{R_g} i_2
\]

\[
\phi_1 = 2N_1 i_1 - \frac{N_2}{R_g} i_2
\]

\[
\lambda_1 = N_1 \phi_1 = 2N_1^2 i_1 - \frac{N_1 N_2}{R_g} i_2 = L_{11} i_1 + L_{12} i_2
\]

\[
\lambda_2 = N_2 \phi_2 = -\frac{N_2}{R_g} i_1 + \frac{N_2}{R_g} i_2 = L_{21} i_1 + L_{22} i_2
\]

\[
L_{11} = \frac{2N_1^2}{R_g} = \frac{2\mu_0 A N_1^2}{g} = \frac{2\mu_0 wd N_1^2}{g}
\]

\[
L_{12} = \frac{L_{21}}{R_g} = -\frac{N_1 N_2}{R_g} = -\frac{\mu_0 A N_1 N_2}{g} = -\frac{2\mu_0 wd N_1 N_2}{g}
\]

\[
L_{22} = \frac{N_2^2}{R_g} = \frac{\mu_0 A N_2^2}{g} = \frac{\mu_0 wd N_2^2}{g}
\]
Problem 5 (15 Points)

(a) Refer to the figure on the left and place a dot on one of the terminals of the secondary coil of the above two-winding transformer.

(b) Place a dot on the corresponding terminal of the schematic representation of this linear transformer on the figure on the right and select the correct polarity signs in the following voltage-current relationships of this two-winding transformer:

\[ v_1(t) = \pm L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \]
\[ v_2(t) = \pm M \frac{di_1}{dt} \pm L_2 \frac{di_2}{dt} \]

Answer:

\[ v_1(t) = + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]
\[ v_2(t) = - M \frac{di_1}{dt} - L_2 \frac{di_2}{dt} \]