Instructions:


2. Work each problem on the exam booklet in the space provided.

3. Write neatly and clearly for partial credit. Cross out any material you do not want graded.

Name: __________________________

Problem 1: _______________________/15

Problem 2: _______________________/20

Problem 3: _______________________/25

Problem 4: _______________________/15

Problem 5: _______________________/25

Total: _______________________/100

Maxwell’s Equations:

Gauss’ Law: \[ \phi_{in} = \phi_{out} \]

Ampere’s Law: \[ \sum_k H_k l_k = I_{encircled} \]

Faraday’s Law: \[ v(t) = \frac{d\lambda}{dt}, \quad \lambda = N\phi = NBA, \quad B = \mu_r \mu_o H, \quad \mu_o = 4\pi \times 10^{-7} \]

Magnetic Circuits:

Ampere’s Law: \[ Ni = Hi \quad \text{or} \quad F = Ni = R\phi, \quad R = \frac{1}{\mu A}, \quad L = \frac{N^2}{R} \]

Faraday’s Law: \[ v(t) = \frac{d\lambda}{dt} = L(x(t))\frac{di}{dt} + i(t)\frac{dL}{dx} \]

Energy and Coenergy: \[ W_m = \int_0^\lambda i \, d\lambda, \quad W_m' = \lambda i - W_m = \int_0^i \lambda \, di \]
Problem 1 (15 Points)

(a) Find the magnetic energy $W_{mA} (J)$ and the magnetic coenergy $W'_{mA} (J)$ at Point A.

\[ W_{mA} = 2 \text{ J} \]

\[ W'_{mA} = 2 \text{ J} \]

Check : \( \lambda_{A}i_{A} = 2 \times 2 = 4 \text{ J} = W_{mA} + W'_{mA} \)

(b) Find the magnetic energy $W_{mB} (J)$ and the magnetic coenergy $W'_{mB} (J)$ at Point B.

\[ W_{mB} = 2 + 1 + 0.5 = 3.5 \text{ J} \]

\[ W'_{mB} = 2 + 4 + 0.5 = 6.5 \text{ J} \]

Check : \( \lambda_{B}i_{B} = 2.5 \times 4 = 10 \text{ J} = W_{mB} + W'_{mB} \)

(c) Compute the apparent inductance $L_{A} = \lambda_{A}/i_{A} (\text{H})$ at Point A and the apparent inductance $L_{B} = \lambda_{B}/i_{B} (\text{H})$ at Point B.

\[ L_{A} = \frac{\lambda_{A}}{i_{A}} = \frac{2}{2} = 1 \text{ H} \]

\[ L_{B} = \frac{\lambda_{B}}{i_{B}} = \frac{2.5}{4} = 0.625 \text{ H} < L_{A} \]
An 100-turn coil is wound around a magnetic core with an air gap as shown above. The air-gap length is \( g = \pi \) (mm). The width and depth are equal to \( w = d = 10 \) (cm). An unknown DC current \( i \) generates a magnetic flux density \( B = 1 \) (T) in the air gap. Assume that the permeability of the magnetic material is infinite and neglect fringing and leakage flux effects.

(a) Find the magnetic energy \( W_m \) (J) stored in the air gap.

\[
W_m = \left( \frac{1}{2} BH \right) (gwd) = \left( \frac{B^2}{2\mu_o} \right) (gwd)
\]
\[
= \frac{1^2 \times \pi \times 10^{-3} \times 10 \times 10^{-2} \times 10 \times 10^{-2}}{2 \times 4\pi \times 10^{-7}} = 12.5 \text{ J}
\]

(b) Find the magnitude of the DC current \( i \) (A).

\[
Ni = Hg \implies i = \frac{Hg}{N} = \frac{Bg}{\mu_o N} = \frac{1 \times \pi \times 10^{-3}}{4\pi \times 10^{-7} \times 100} = 25 \text{ A}
\]

or \( Ni = R_g \phi \implies i = \frac{R_g \phi}{N} = \frac{gBwd}{\mu_o wdN} = \frac{gB}{\mu_o N} = \frac{\pi \times 10^{-3} \times 1}{4\pi \times 10^{-7} \times 100} = 25 \text{ A} \)

(c) Find the coil inductance \( L \) (mH) using the magnetic energy \( W_m \) in the air gap.

\[
W_m = W_m' = \frac{1}{2} Li^2 \implies L = \frac{2W_m}{i^2} = \frac{2 \times 12.5}{25^2} = 0.04 \text{ mH}
\]

Check:

\[
L = \frac{N^2}{R_g} = \frac{\mu_o wdN^2}{g} = \frac{4\pi \times 10^{-7} \times 0.1 \times 0.1 \times 100^2}{\pi \times 10^{-3}} = 40 \text{ mH}
\]

(d) Find the magnetic flux linkages \( \lambda \) (Wb-t) of the coil.

\[
\lambda = Li = 0.04 \times 25 = 1 \text{ Wb-t}
\]

or \( \lambda = N\phi = NBA = 100 \times 1 \times 0.1 = 1 \text{ Wb-t} \)

(e) Check that the magnetic energy of the coil is equal to \( 0.5\lambda i \) (J).

\[
\frac{1}{2} \lambda i = \frac{1}{2} \times 1 \times 25 = 12.5 \text{ J} = W_m
\]
Problem 3 (25 Points)

The magnetic material in the above structure has the corresponding B-H curve shown on the right. The core has a mean length \( l = 20 \text{ cm} \) and a cross-sectional area \( A = 50 \text{ cm}^2 \). The coil has \( N = 100 \) turns and a resistance \( R = 4 \Omega \). The coil is energized with a DC voltage \( E = 30 \text{ V} \). Neglect all leakage fluxes.

(a) Find the DC current \( I \) into the coil in A.

\[
I = \frac{E}{R} = \frac{30}{4} = 7.5 \text{ A}
\]

(b) Find the magnetic field intensity \( H \) in the magnetic material in A-t/m.

\[
Hl = Ni \implies H = \frac{Ni}{l} = \frac{100 \times 7.5}{0.2} = 3750 \text{ A-t/m}
\]

(c) Find the magnetic flux density \( B \) in the magnetic core in T.

\( B = 2 \text{ T} \) (read from the B-H curve)

(d) Find the magnetic flux \( \phi \) in the core in mWb.

\[
\phi = BA = 2 \times 50 \times 10^{-4} = 10 \text{ mWb}
\]

(e) Find the flux linkages \( \lambda \) with the N-turn coil in Wb-t.

\[
\lambda = N\phi = 100 \times 0.01 = 1 \text{ Wb-t}
\]

(f) Find the stored magnetic energy \( W_m \) in J. Is it equal to \( 0.5\lambda i \) (J)?

\[
W_m = \left( \frac{1}{2}B_{\text{max}}H_{\text{max}} \right)(Al) = \left( \frac{1}{2} \times 2 \times 2500 \right) \left( 50 \times 10^{-4} \times 20 \times 10^{-2} \right) = 2.5 \text{ J}
\]

\[
\frac{1}{2}\lambda i = \frac{1}{2} \times 1 \times 7.5 = 3.75 \text{ J} \neq W_m
\]
Problem 4 (15 Points)

A coil with a movable part is characterized by the following flux-current relationship,

\[ \lambda = L(x)i = \frac{ki}{x/x_o} \]

where \( x \) is a linear motion variable and \( k \) and \( x_o \) are constants. If the coil is excited with a constant current \( i(t) = I \) and the motion variable varies sinusoidally as \( x(t) = 2x_o + x_o \sin \omega t \), find the voltage \( v(t) \) induced at the terminals of the coil as a function of time \( t \).

\[
\begin{align*}
\lambda(t) &= L(x(t))i(t) \\
v(t) &= \frac{d\lambda}{dt} = L(x(t))\frac{di}{dt} + i(t)\frac{dL(x(t))}{dt} \\
&= L(x(t))\frac{di}{dt} + i(t)\frac{dL}{dx}\frac{dx}{dt} \\
&= \left( \frac{kx_o}{x(t)} \right) \frac{dI}{dt} + I \left( -\frac{kx_o}{x^2(t)} \right) \frac{dx}{dt} \\
&= \left( \frac{kx_o}{2x_o + x_o \sin \omega t} \right) (0) + I \left[ -\frac{kx_o}{(2x_o + x_o \sin \omega t)^2} \right] (\omega x_o \cos \omega t) \\
&= \frac{-\omega k x_o^2 I \cos \omega t}{(2x_o + x_o \sin \omega t)^2} \\
&= \frac{-\omega k I \cos \omega t}{(2 + \sin \omega t)^2}
\end{align*}
\]

Alternate Solution:

\[
\begin{align*}
\lambda(t) &= L(x(t))i(t) = \frac{kx_o}{x(t)}i(t) = \frac{kx_o I}{2x_o + x_o \sin \omega t} = \frac{kI}{2 + \sin \omega t} \\
v(t) &= \frac{d\lambda}{dt} = -\frac{\omega kI \cos \omega t}{(2 + \sin \omega t)^2}
\end{align*}
\]
Neglect leakage and fringing effects in this problem. The magnetic core is assumed to have infinite permeability. The coils are characterized by the following flux-current relationships:

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} = \begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

(a) Add polarity marks to the mmfs on the equivalent magnetic circuit representation of this magnetic structure. Specify the reluctance \( R_g \) of the air gap as a function of \( \mu_0, g \), and \( A = wd \).

(b) Solve for \( \phi, \lambda_1, \) and \( \lambda_2 \), and deduce \( L_{11}, L_{12}, L_{21}, \) and \( L_{22} \) in terms of \( N_1, N_2 \) and \( R_g \).

\[
\phi = \frac{N_1i_1 - N_2i_2}{R_g}
\]

\[
\lambda_1 = N_1\phi_1 = N_1\phi = \left( \frac{N_2^2}{R_g} \right) i_1 - \left( \frac{N_1N_2}{R_g} \right) i_2 = L_{11}i_1 + L_{12}i_2
\]

\[
\lambda_2 = N_2\phi_2 = -N_2\phi = -\left( \frac{N_2N_1}{R_g} \right) i_1 + \left( \frac{N_2^2}{R_g} \right) i_2 = L_{21}i_1 + L_{22}i_2
\]

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2
\end{bmatrix} = \begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2
\end{bmatrix} = \frac{1}{R_g} \begin{bmatrix}
N_1^2 & -N_1N_2 \\
-N_2N_1 & N_2^2
\end{bmatrix} \begin{bmatrix}
i_1 \\
i_2
\end{bmatrix}
\]

(c) If the two coupled coils are connected in series as shown by the dashed line, find the effective inductance of the series coil connection.

\[
\lambda = \lambda_1 - \lambda_2 = (L_{11}i_1 + L_{12}i_2) - (L_{21}i_1 + L_{22}i_2) = (L_{11} + L_{22} - L_{12} - L_{21})i
\]

\[
L = \frac{\lambda}{i} = \frac{N_1^2}{R_g} + \frac{N_2^2}{R_g} + \frac{N_1N_2}{R_g} + \frac{N_2N_1}{R_g} = \frac{N_1^2 + N_2^2 + 2N_1N_2}{R_g}
\]

\[
= \frac{(N_1 + N_2)^2}{R_g}
\]